Quantum deformation of KdV hierarchies and their infinitely many conservation laws

1993 J. Phys. A: Math. Gen. 26 2389

(http://iopscience.iop.org/0305-4470/26/10/011)

J. Phys. A: Math. Gen. 26 (1993) 2389–2408. Printed in the UK

Quantum deformation of KdV hierarchies and their infinitely many conservation laws

De-Hai Zhang

Center of Theoretical Physics, CCAST (World Laboratory), Beijing, People's Republic of China, and Physics Department, Graduate School, Academia Sinica, PO Box 3908, Beijing 100039, People's Republic of China

Received 23 April 1992, in final form 11 December 1992

Abstract. A reasonable q-deformed differential is defined. A set of operation rules are constructed for the q-deformed formal pseudo differential operators. The complete procedure of constructing the q-deformed KdV hierarchies and their infinite conservation laws is given. As an important example, we obtain a detailed structure of the simplest (3,2) system, i.e. the q-deformed KdV equations.

1. Introduction

Recently, the (1+1)-dimensional Korteweg-de Vries (KdV) hierarchy [1] has attracted considerable attention among theoretical physicists. The results of research demonstrate that the KdV hierarchy is closely related to the following popular topics: (1) matrix models [2] and non-perturbative treatment of 2-dimensional field theories [3], (2) theories of 2-dimensional gravity coupled to matter systems [4], (3) 2-dimensional topological field theories [5] and (4) conformal field theory [6] and W algebras [7]. The basic equations governing non-perturbative 2-dimensional gravity coupled to minimal models are the differential equations of KdV hierarchy. The partition function and the correlation functions of the 2-dimensional topological gravity coupled to minimal models are conjectured to be described by the KdV hierarchy. The KdV hierarchy shows the miraculous power and mysterious relations in treatment of different mathematics and physics objects.

On the other hand, the interest to the quantum deformation (so-called q-deformation) of Lie algebra (the quantum group) has been growing in the physical and mathematical regions [8]. The idea of quantum Lie algebras originated from the study of the solution of the quantum Yang–Baxter equation for the integrable lattice models [9]. The representation theory of the q-deformed simple Lie algebras has been investigated widely [10]. One of the methods which is well worth our attention in the study of quantum groups is the q-harmonic oscillator realization of quantum groups [11]. Several authors have extended the definition of q-differentiation [12–13].

The success of quantum groups stimulates people to look for new objects which can perform the analogous so-called q-deformation. For example, the q-deformed Virasoro algebra has been studied in [14] [15] and [16]. Chaichian *et al* even researched the q-deformed KdV system [17]. However, it is hard to say that all of these attempts has been accomplished perfectly.

It was always significant if a new kind of integrable system could be discovered. In this paper we shall conduct a new investigation about q-deformed KdV hierarchies by defining a suitable q-differentiation. Through building a set of complete operations of the q-deformed formal pseudo-differential operator and using the Lax pair, we present the constructing program of q-deformed KdV hierarchies and obtain a q-deformed generalization of the ordinary KdV equation. We conjecture that the q-deformed KdV hierarchies thus produced is a new kind of integrable system.

2. The q-deformed formal pseudo-differential operator

At first we introduce two operators, Q and $\widetilde{Q}$, which are defined as

$$Qf(z) = f(zq) \tag{2.1}$$
$$\widetilde{Q}f(z) = q^{z\partial} f(z) \tag{2.2}$$

where $\partial = \frac{\partial}{\partial z}$ and q is called a deformed parameter. To avoid complexity, q is limited to be a real parameter which is not -1. One can rewrite the operator Q as a formal differential operator with infinite order of the ordinary differential operator ∂,

$$Qf(z) = f(z - \epsilon z) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\epsilon z)^n \partial^n f(z) \tag{2.3}$$

therefore

$$Q = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \epsilon^n z^n \partial^n \tag{2.4}$$

where

$$\epsilon = 1 - q. \tag{2.5}$$

The operator $\widetilde{Q}$ can also be rewritten as an infinite order differential operator

$$\widetilde{Q} = \exp[z\partial \ln(1-\epsilon)] = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \left(\sum_{m=1}^{\infty} \frac{1}{m}\epsilon^m\right)^n (z\partial)^n. \tag{2.6}$$

One can prove that the operator $\widetilde{Q}$ is indeed same as the operator Q,

$$\widetilde{Q} = Q. \tag{2.7}$$

This proof is given in appendix A. The commutative relation between z and Q (or $\widetilde{Q}$ without distinction) is described by the q-deformed commutator

$$[Q, z]_q = 0 \tag{2.8}$$

where

$$[A, B]_r = AB - rBA. \tag{2.9}$$

When $r = 1$, we omit this 1 and denote it simply as

$$[A, B] = AB - BA \tag{2.10a}$$

and when $r = -1$ it becomes the ordinary anticommutator

$$\{A, B\} = AB + BA. \tag{2.10b}$$

After owning the infinite-order differential operator Q, we define the q-deformed differential operator [18]

$$\tilde{D} = \frac{1}{(1-q^{-2})z}(1 - Q^{-2}) \tag{2.11}$$

which is also a formal differential operator of infinite order. It is easy to see that $\tilde{D}$ tends to ∂ when q tends to 1:

$$\lim_{q \to 1} \tilde{D} = \partial. \tag{2.12}$$

The commutative relation between Q and $\tilde{D}$ is also described by the q-deformed commutator

$$[\tilde{D}, Q]_q = 0. \tag{2.13}$$

According to the definition (2.11) of $\tilde{D}$, one can prove the q-deformed Leibniz rule

$$\tilde{D}(f(z)g(z)) = (\tilde{D}f(z))g(z) + (Q^{-2}f(z))(\tilde{D}g(z)) \tag{2.14}$$

which can be expressed in an operator form

$$\tilde{D} \circ f = f^{(0,-2)}\tilde{D} + f^{(1,0)}. \tag{2.15}$$

Here $\circ$ represents a fact that $\tilde{D}$ before $\circ$ must acts on the other functions behind $f(z)$. In the above formulas one introduces the symbol

$$f^{(n,m)}(z) = (\tilde{D}^n Q^m f(z)) \tag{2.16}$$

where $\tilde{D}$ and Q in the parentheses do not act on the functions behind $f(z)$. For example, one has $z^{(1,0)} = 1$. Using the formula (2.15) one gets the q-deformed commutator between the q-deformed differential operator $\tilde{D}$ and the coordinate variable z,

$$[\tilde{D}, z]_{q^{-2}} = 1. \tag{2.17}$$

Applying the q-deformed Leibniz rule (2.15) to the high-order case of the q-deformed differential operator, we obtain

$$\tilde{D}^n \circ f(z) = \sum_{m=0}^{n} \left[\binom{n}{m}\right] q^{2m(n-m)} f^{(m,2m-2n)}(z)\tilde{D}^{n-m} \tag{2.18}$$

where

$$\left[\binom{n}{m}\right] = \frac{[n]!}{[m]![n-m]!} \tag{2.19}$$

$$[m]! = [m][m-1]\cdots[2][1] \tag{2.20}$$

$$[m] = \frac{1-q^{-2m}}{1-q^{-2}} = q^{1-m}\lfloor m \rfloor \tag{2.21}$$

and

$$[0]! = 0! = 1. \tag{2.22}$$

In (2.21), the symbol

$$\lfloor m \rfloor = (q^m - q^{-m})(q - q^{-1})^{-1} \tag{2.23}$$

is just the ordinary q-deformed one, often used in references. One can generalize (2.18) to the case of negative n,

$$\tilde{D}^{-n} \circ f(z) = \sum_{m=0}^{\infty} (-1)^m \left[\binom{n+m-1}{m} \right] q^{-m(m+1)} f^{(m,2m+2n)}(z) \tilde{D}^{-n-m}. \tag{2.24}$$

Specially for the case of $n = -1$, one has

$$\tilde{D}^{-1} \circ f = f^{(0,2)} \tilde{D}^{-1} - q^{-2} f^{(1,4)} \tilde{D}^{-2} + q^{-6} f^{(2,6)} \tilde{D}^{-3} - \cdots . \tag{2.25}$$

It is necessary to note that q-deformed differential operators differ from the ordinary difference operators. Although the q-deformed differential operator seems a difference operator at the level of the first order form, the second order q-deformed differential operator

$$\tilde{D}^2 f(z) = \frac{f(z) - (1+q^2) f(zq^{-2}) + q^2 f(zq^{-4})}{(1-q^{-2})^2 z^2} \tag{2.26}$$

differs from a second-order difference operator, due to its equal-ratio distances between points and its non-standard coefficients. In fact the q-deformed differential operator is called the q-difference. Up to now we have described all of operations needed to construct q-deformed KdV hierarchies.

A q-deformed KdV hierarchy is described in terms of the q-deformed pseudo-differential operator which is given by a formal expression

$$K = \sum_{n=-\infty}^{M} k_n \tilde{D}^n \tag{2.27}$$

where the coefficients are functions $k_n(z)$ in a variable z and $\tilde{D}$ is defined as (2.11). The multiplicative rule of two q-deformed pseudo-differential operators has been given by formulas (2.18) and (2.24). One further introduces the decomposition

$$K_+ = \sum_{n=0}^{M} k_n \tilde{D}^n \tag{2.28}$$

$$K_- = \sum_{n=1}^{\infty} k_{-n} \tilde{D}^{-n} \tag{2.29}$$

$$\widetilde{\mathrm{res}} K = k_{-1} \tag{2.30}$$

where '$\widetilde{\mathrm{res}}$' stands for the q-deformed residue.

3. The q-deformed KdV hierarchy and the q-deformed KdV equation

The Nth q-deformed KdV hierarchy consists of an infinite set of commuting q-deformed differential equations for the coefficients $V_n(z, t_p)$ $(n = 0, 1, \cdots, N-1)$ of a q-deformed differential operator L of order N that has been put in the canonical form

$$L = \tilde{D}^N + \sum_{n=0}^{N-1} V_n \tilde{D}^n. \tag{3.1}$$

In the algebra of q-deformed pseudo-differential operators L has an unique Nth root $L^{1/N}$, and in the Lax representation [19] the pth flow of the Nth q-deformed KdV hierarchy (called the (p, N) system) is given by

$$\frac{\partial}{\partial t_p} L = [L_+^{p/N}, L] \tag{3.2}$$

where t_p are called time parameters. Since L commutes with $L^{p/N}$, one has

$$[L_+^{p/N}, L] = [L, L_-^{p/N}]. \tag{3.3}$$

But since from the LHS above, the commutator can have only positive powers of $\tilde{D}$, and since from the RHS above the highest-order term is only up to one of $\tilde{D}^{N-1}$, the expression (3.3) is only an order-$N-1$ q-deformed differential operator without negative powers of $\tilde{D}$. When expanded in powers of $\tilde{D}$ this operator equation (3.2) gives rise to a single q-deformed differential equation for each of the coefficients V_n. The Lax pair structure of q-deformed KdV hierarchies is the first indication that they may be completely integrable systems.

The simplest system of the (p, N) q-deformed KdV hierarchies must be the (3,2) system which is known as q-deformed KdV equations. Let us take this system in order to illustrate the above procedure. This model is obtained by taking L to be the second-order q-deformed differential operator

$$K = \tilde{D}^2 + V_1(z, t)\tilde{D} + V_0(z, t). \tag{3.4}$$

The formal expansion of $L^{1/2}$ in powers of D is given by

$$K^{1/2} = \tilde{D} + \sum_{n=0}^{\infty} W_{-n} \tilde{D}^{-n}. \tag{3.5}$$

Since one needs only the first five coefficients of W_{-n} in the later q-deformed KdV equations, one gives them in terms of V_1 and V_0 order by order,

$$W_0 = (1 + Q^{-2})^{-1} V_1 = \sum_{n=0}^{\infty} (-1)^n V_1^{(0,-2n)} \tag{3.6}$$

$$W_{-1} = -(1 + Q^{-2})^{-1}(-V_0 + W_0^{(1,0)} + W_0) \tag{3.7}$$

$$W_{-2} = -(1 + Q^{-2})^{-1}(W_{-1} W_0^{(0,2)} + W_{-1}^{(1,0)} + W_0 W_{-1}) \tag{3.8}$$

$$\begin{aligned} W_{-3} = -(1 + Q^{-2})^{-1}(&-q^{-2} W_{-1} W_0^{(1,4)} + W_{-2} W_0^{(0,4)} + W_{-1} W_{-1}^{(0,2)} \\ &+ W_{-2}^{(1,0)} + W_0 W_{-2}) \end{aligned} \tag{3.9}$$

$$\begin{aligned} W_{-4} = -(1 + Q^{-2})^{-1}(&q^{-6} W_{-1} W_0^{(2,6)} - q^{-2}[2] W_{-2} W_0^{(1,6)} + W_{-3} W_0^{(0,6)} \\ &- q^{-2} W_{-1} W_{-1}^{(1,4)} + W_{-2} W_{-1}^{(0,4)} + W_{-1} W_{-2}^{(0,2)} \\ &+ W_{-3}^{(1,0)} + W_0 W_{-3}). \end{aligned} \tag{3.10}$$

The first formula of (3.6) is regarded as one of the methods to calculate operator $(1+Q^{-2})^{-1}$. The q-deformed differential operator needed in the (3,2) system is $L_+^{3/2}$. Owing to the identity (3.3), one needs only the first two terms of the q-deformed pseudo differential operator $L_-^{3/2}$,

$$K_-^{3/2} = U_{-1}\tilde{D}^{-1} + U_{-2}\tilde{D}^{-2} + \cdots \tag{3.11}$$

where the coefficients U_{-1} and U_{-2} are given by the expressions

$$U_{-1} = W_{-3} - q^{-2}W_{-1}V_1^{(1,4)} + W_{-2}V_1^{(0,4)} + W_{-1}V_0^{(0,2)} \tag{3.12}$$

$$U_{-2} = W_{-4} + q^{-6}W_{-1}V_1^{(2,6)} - q^{-2}[2]W_{-2}V_1^{(1,6)} + W_{-3}V_1^{(0,6)} - q^{-2}W_{-1}V_0^{(1,4)} + W_{-2}V_0^{(0,4)}. \tag{3.13}$$

Now we can obtain the q-deformed KdV equations by using equations (3.2), (3.3) and (3.11),

$$\frac{\partial V_1}{\partial t} = U_{-1}^{(0,-4)} - U_{-1} \tag{3.14}$$

$$\frac{\partial V_0}{\partial t} = (U_{-2}^{(0,-4)} - U_{-2}) + V_1(U_{-1}^{(0,-2)} - U_{-1}) - U_{-1}(V_1^{(0,-2)} - V_1) + q^2[2]U_{-1}^{(1,-2)}. \tag{3.15}$$

By using equations (3.12)–(3.13) and (3.6)–(3.10), the right sides of the equations (3.14) and (3.15) can be finally expressed in terms of pure V_1 and V_0. Because the q-deformed differential operators are not the ordinary difference operators, the q-deformed KdV equations are not the ordinary differencing of the ordinary KdV equation.

4. The expanding expression of the q-deformed differential operators

Usually we know well the ordinary differential operators and are not familiar with the q-deformed differential operators. In this section our main task is to present various kinds of q-deformed operators in terms of ordinary differential operators. From the definition (2.1) of Q, we have

$$Q^m f(z) = f(z + (q^m - 1)z) = \sum_{n=0}^{\infty} \frac{1}{n!}(q^m - 1)^n z^n \partial^n f(z). \tag{4.1}$$

We therefore obtain

$$Q^m = \sum_{n=0}^{\infty} \frac{1}{n!}(q^m - 1)^n z^n \partial^n. \tag{4.2}$$

As for the operator $\tilde{D}$, according to (2.11), one gets

$$\tilde{D}^m = \left(\frac{1}{(1-q^{-2})z}(1 - Q^{-2})\right)^m \tag{4.3}$$

which can be expanded as

$$\tilde{D}^m = \frac{1}{(1-q^{-2})^m z^m} \prod_{i=0}^{m-1} (1 - q^{2i} Q^{-2}) \tag{4.4}$$

$$= \frac{1}{(1-q^{-2})^m z^m} \sum_{j=0}^{n} (-1)^j \left[\binom{m}{j} \right] q^{j(2m-j-1)} Q^{-2j} \tag{4.5}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{m} (-1)^j \left[\binom{m}{j} \right] \frac{q^{j(2m-j-1)}}{(1-q^{-2})^m} (q^{-2j} - 1)^n \right) \frac{1}{n!} z^{n-m} \partial^n \tag{4.6}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^{k+n} [k+m]!}{(k+m)!(n-k)![k]!} \right) z^n \partial^{n+m} \qquad (m \geqslant 0) \tag{4.7}$$

and finally one obtains

$$\tilde{D}^m = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^{k+n} \widetilde{\Gamma}(k+m+1)}{\Gamma(k+m+1)(n-k)![k]!} \right) z^n \partial^{n+m}. \tag{4.8}$$

Here we have introduced the ordinary Gamma function

$$\Gamma(n+1) = n\Gamma(n) \tag{4.9}$$

$$\Gamma(n+1) = n! \qquad (n \geqslant 0) \tag{4.10}$$

$$\Gamma(-n) = \frac{(-1)^n}{n!} \Gamma(0) \qquad (n \geqslant 0) \tag{4.11}$$

and a q-deformed Gamma function

$$\widetilde{\Gamma}(n+1) = [n] \widetilde{\Gamma}(n) \tag{4.12}$$

$$\widetilde{\Gamma}(n+1) = [n]! \qquad (n \geqslant 0) \tag{4.13}$$

$$\widetilde{\Gamma}(-n) = \frac{(-1)^n}{[n]!} q^{-n(n+1)} \widetilde{\Gamma}(0) \qquad (n \geqslant 0) \tag{4.14}$$

where the $\Gamma(0)$ is an infinity which is a formal symbol and the $\widetilde{\Gamma}(0)$ is to be determined. Defining

$$\Gamma_q = \frac{\widetilde{\Gamma}(0)}{\Gamma(0)} \tag{4.15}$$

one has

$$\frac{\widetilde{\Gamma}(-n+1)}{\Gamma(-n+1)} = \frac{(n-1)!}{[n-1]!} q^{-n(n-1)} \Gamma_q \qquad (n \geqslant 1). \tag{4.16}$$

The above formulae are ready for extending the expansion (4.8) to the case of negative powers of the q-deformed differential operator. We find

$$\tilde{D}^{-m} = \left(\sum_{n=0}^{m-1}\sum_{k=0}^{n} + \sum_{n=m}^{\infty}\sum_{k=0}^{m-1} + \sum_{n=m}^{\infty}\sum_{k=m}^{n}\right) \frac{(-1)^{k+n}\widetilde{\Gamma}(k-m+1)}{\Gamma(k-m+1)(n-k)![k]!} z^n \partial^{n-m} \tag{4.17}$$

$$= \left(\sum_{n=0}^{m-1}\sum_{k=0}^{n} + \sum_{n=m}^{\infty}\sum_{k=0}^{m-1}\right) \frac{(-1)^{k+n}(m-k-1)!q^{-(m-k)(m-k-1)}\Gamma_q}{[m-k-1]!(n-k)![k]!} z^n \partial^{n-m}$$
$$+ \sum_{n=m}^{\infty}\sum_{k=m}^{n} \frac{(-1)^{k+n}[k-m]!}{(k-m)!(n-k)![k]!} z^n \partial^{n-m}. \tag{4.18}$$

A task is to determine the quantity Γ_q. Let us inspect a simple case

$$\tilde{D} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \omega^n z^n \partial^{n+1} \tag{4.19}$$

where

$$\omega = q^{-2} - 1. \tag{4.20}$$

After moving the differential operator from right to left side, we obtain

$$\tilde{D} = \partial \left(\sum_{n=1}^{\infty} a_n \frac{1}{n!} z^n \partial^n + a_0\right) \tag{4.21}$$

where

$$a_n = \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{n+k}}{n+k+1} \tag{4.22}$$

and

$$a_0 = \sum_{k=0}^{\infty} \frac{(-1)^k \omega^k}{k+1} = \frac{\ln(1+\omega)}{\omega}. \tag{4.23}$$

It is easy to obtain the inverse of $\tilde{D}$ from (4.21):

$$\tilde{D}^{-1} = \left(1 + \sum_{n=1}^{\infty} \frac{a_n}{n! a_0} z^n \partial^n\right)^{-1} a_0^{-1} \partial^{-1}. \tag{4.24}$$

Comparing the above result with the general expansion (4.18), one finds

$$\Gamma_q = a_0^{-1} = \frac{1-q^{-2}}{2\ln q}. \tag{4.25}$$

After getting the expression for expansion in terms of the ordinary differential operators, one can take the ordinary residue

$$\operatorname{res} B = b_{-1} \tag{4.26}$$

for an ordinary formal pseudo differential operator with infinite order at both limits:

$$B = \sum_{n=-\infty}^{\infty} b_n \partial^n. \tag{4.27}$$

According to this definition of the ordinary residue and the results of (4.18), one can obtain

$$\mathrm{res}(\tilde{D}^{-m}) = \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} q^{-(m-k)(m-k-1)} \Gamma_q}{[k]![m-k-1]!} z^{m-1}. \tag{4.28}$$

5. The infinite conservation laws

One of the most important properties of some integrable systems is that they possess infinitely many conservation laws. We shall prove in this section that the q-deformed KdV hierarchies have infinitely many conservation laws. Our method comes from the one of Drinfeld and Sokolov. Since the cases are very similar, the reader can refer to [20] for details. We can prove that the flows determined by the Lax equations commute with one another. If A is a q-deformed differential operator which satisfies the Lax pair equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [A, L] \tag{5.1}$$

where L is a q-deformed differential operator (3.1), then one has

$$\frac{\mathrm{d}}{\mathrm{d}t} L^{r/k} = [A, L^{r/k}]. \tag{5.2}$$

Let us consider the equations

$$\frac{\partial L}{\partial t} = [M_+, L] \qquad M = \sum c_i L^{i/k} \tag{5.3}$$

and

$$\frac{\partial L}{\partial \tau} = [\tilde{M}_+, L] \qquad \tilde{M} = \sum \tilde{c}_i L^{i/k}. \tag{5.4}$$

It can be verified that

$$\frac{\partial L}{\partial t \partial \tau} = \frac{\partial L}{\partial \tau \partial t} \tag{5.5}$$

which demonstrates the consistency of the q-deformed KdV hierarchies. This proof is given in appendix B.

Drinfeld and Sokolov have pointed out that if P and Q are formal ordinary pseudo differential operators, then $\mathrm{res}[P, Q]$ is a total derivative of some differential polynomial in the coefficients of P and Q. This conclusion is also suitable for the q-deformed formal differential operators, since they can be expanded as series with positive and negative infinity orders of ordinary differential operator

$$P = \sum_{m=-\infty}^{\infty} \tilde{a}_m \partial^m \qquad Q = \sum_{m=-\infty}^{\infty} \tilde{b}_l \partial^l. \tag{5.6}$$

According to the result of Drinfeld and Sokolov one then gets

$$\mathrm{res}[P, Q] = \frac{\partial g}{\partial z} \tag{5.7}$$

where

$$g = \sum_{m,\, l=-\infty}^{\infty} \frac{m(m-1)\cdots(1-l)(-l)}{(m+l+1)!} \left\{ \sum_{i=0}^{m+l} (-1)^i \, \tilde{a}_m^{(i)} \, \tilde{b}_l^{(m+l-i)} \right\} \tag{5.8}$$

$$\tilde{a}^{(i)} = (\partial^i \tilde{a}). \tag{5.9}$$

Taking the ordinary residue on both sides of equation (5.2) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{res} L^{r/k} = \mathrm{res}[A, L^{r/k}] = \frac{\partial f}{\partial z}. \tag{5.10}$$

Integrating it and choosing suitable boundary condition one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}z \, \mathrm{res} L^{r/k} = \int \frac{\partial f}{\partial z} \mathrm{d}z = 0. \tag{5.11}$$

Therefore we see that for any integer r, the residue

$$H_{r/k} = \mathrm{res} L^{r/k} \tag{5.12}$$

is a density of conservation law for the Lax equation. Of course, nontrivial conservation laws

$$C_{r/k} = \int \mathrm{d}z \, \mathrm{res} L^{r/k} \tag{5.13}$$

correspond only to numbers r not a multiple of k. If one knows the expansion expression of the flows in terms of the q-deformed differential operators

$$L^{r/k} = \sum_{n=-\infty}^{r} a_{r-n} \tilde{D}^n \tag{5.14}$$

one can obtain from equation (4.18) the density of the conservation law in terms of the coefficients of the expansion expression

$$H_{r/k} = \mathrm{res}\, L^{r/k} = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} \frac{(-1)^{n-i-1} q^{-(n-i)(n-i-1)} \Gamma_q}{[i]![n-i-1]!} \right) a_{r+n} z^{n-1}. \tag{5.15}$$

The existence of infinitely many conservation laws of the q-deformed KdV hierarchies is the second indication that they may be completely integrable systems.

An important example is for the q-deformed differential operator of order two

$$L = K = \tilde{D} + V_1 \tilde{D} + V_0. \tag{5.16}$$

We hope to obtain the recursion formulas of the so-called q-deformed Gelfand–Dikii potentials R_l and S_l, which are relevant to the nontrivial densities of conservation laws of the first order formalism and defined as

$$K_-^{l-\frac{1}{2}} = R_l \tilde{D}^{-1} + S_l \tilde{D}^{-2} + \cdots. \tag{5.17}$$

From the obvious identity

$$[K^{l-\frac{1}{2}}, K] = 0 \tag{5.18}$$

one gets

$$[K_+^{l-\frac{1}{2}}, K] = [K, K_-^{l-\frac{1}{2}}]_+ \tag{5.19}$$

$$= (R_l^{(0,-4)} - R_l)\tilde{D} + (q^2[2]R_l^{(1,-2)} + S_l^{(0,-4)} - S_l + V_1 R_l^{(0,2)} - R_l V_1^{(0,2)}). \tag{5.20}$$

On the other hand one has

$$K_+^{l+\frac{1}{2}} = \tfrac{1}{2}\{K_+^{l-\frac{1}{2}}, K\} + \tfrac{1}{2}\{K_-^{l-\frac{1}{2}}, K\}_+. \tag{5.21}$$

Having performed

$$[K_+^{l+\frac{1}{2}}, K] = \tfrac{1}{2}\{[K_+^{l-\frac{1}{2}}, K], K\} + \tfrac{1}{2}[\{K_-^{l-\frac{1}{2}}, K\}_+, K] \tag{5.22}$$

we obtain

$$\begin{aligned} R_{l+1} = (Q^{-4} - 1)^{-1}\{&(1 - q^2[2])R_l^{(0,-4)} V_1^{(1,0)} \\ &+ R_l^{(0,-4)} V_0^{(0,-2)} - R_l V_0 - R_l^{(2,0)} - R_l^{(1,0)} V_1 \\ &+ S_l^{(0,-4)} V_1 - S_l^{(0,-2)} V_1 - q^2[2] S_l^{(1,-2)}\} \end{aligned} \tag{5.23}$$

$$\begin{aligned} S_{l+1} = (Q^{-4} - 1)^{-1}\{&-q^2[2]R_{l+1}^{(1,-2)} + R_{l+1} V_1^{(0,2)} - R_{l+1}^{(0,-2)} V_1 \\ &+ R_l^{(0,-4)} V_0^{(1,0)} - R_l^{(0,-4)} V_1^{(2,2)} + q^2[2] R_l^{(1,-2)}(V_0 - V_1^{(1,-2)}) \\ &- R_l^{(2,0)} V_1^{(0,2)} + R_l^{(0,-2)} V_1 V_0 - R_l V_1^{(0,2)} V_0 - R_l^{(1,0)} V_1^{(0,2)} V_1 \\ &- R_l^{(0,-2)} V_1^{(1,2)} V_1 - S_l^{(2,0)} + S_l^{(0,-4)} V_0 - S_l V_0 - S_l^{(1,0)} V_1\} \end{aligned} \tag{5.24}$$

which are just the recursion formulas for R_l and S_l with the initial values

$$R_0 = 1 \qquad S_0 = -W_0^{(0,2)} \tag{5.25}$$

where W_0 is given in equation (3.6). Using these recursion formulas for the case $l = 0$ one can again obtain the results

$$R_1 = W_{-1} \qquad \text{and} \qquad S_1 = W_{-2} \tag{5.26}$$

where W_{-1} and W_{-2} have been given in equations (3.7) and (3.8).

6. First-order formalism of q-deformed KdV equation

The form (3.14-15) of the q-deformed KdV equations is difficult to understood. In order to compare it with the ordinary KdV equation, we must inspect the difference between the ordinary KdV equation and its q-deformed version when the deformed parameter q tends to 1. Letting $q = 1 - \epsilon$, up to the second order of infinitesimal parameter ϵ, one has from (4.2) that

$$Q^n = 1 - \epsilon n z\partial + \frac{\epsilon}{2}(n(n-1)z\partial + n^2 z^2 \partial^2) + O(\epsilon^3) \tag{6.1}$$

and from (4.8) up to the first order:

$$\tilde{D}^m = \partial^m + \epsilon(mz\partial^{m+1} + \tfrac{1}{2}m(m-1)\partial^m) + O(\epsilon^2). \tag{6.2}$$

Then one obtains

$$\tilde{D}^m Q^n = \partial^m + \epsilon\left(\tfrac{1}{2}m(m-2n-1)\partial^m + (m-n)z\partial^{m+1}\right) + O(\epsilon^2). \tag{6.3}$$

Since the first equation (3.14) of the q-deformed KdV equations becomes

$$\frac{\partial V_1}{\partial t} = (Q^{-4} - 1)U_{-1} = 4\epsilon z U'_{-1} \tag{6.4}$$

one learns that V_1 is a quantity of the same order as ϵ. Using (6.3) one can simplify the relations (3.6-10) and (3.12-13) up to the first order in ϵ

$$W_0 = \tfrac{1}{2}V_1 + O(\epsilon^2) \tag{6.5}$$
$$W_{-1} = \tfrac{1}{2}V_0 - \tfrac{1}{4}V_1' - \tfrac{1}{2}\epsilon z V_0' + O(\epsilon^2) \tag{6.6}$$
$$W_{-2} = -\tfrac{1}{4}V_0' - \tfrac{1}{4}V_0 V_1 + \tfrac{1}{8}V_1'' + \tfrac{1}{4}\epsilon V_0' + \tfrac{1}{4}\epsilon z V_0'' + O(\epsilon^2) \tag{6.7}$$
$$\begin{aligned} W_{-3} = &-\tfrac{1}{8}V_0^2 + \tfrac{1}{8}V_0'' + \tfrac{3}{8}V_0 V_1' + \tfrac{1}{4}V_1 V_0' - \tfrac{1}{16}V_1''' \\ &+ \tfrac{3}{4}\epsilon z V_0 V_0' - \tfrac{1}{4}\epsilon V_0'' - \tfrac{1}{8}\epsilon z V_0''' + O(\epsilon^2) \end{aligned} \tag{6.8}$$
$$W_{-4} = \tfrac{3}{8}V_0 V_0' - \tfrac{1}{16}V_0''' + O(\epsilon) \tag{6.9}$$

and

$$U_{-1} = \tfrac{3}{8}V_0^2 + \tfrac{1}{8}V_0'' - \tfrac{3}{8}V_0 V_1' - \tfrac{3}{4}\epsilon z V_0 V_0' - \tfrac{1}{16}V_1''' - \tfrac{1}{4}\epsilon V_0'' - \tfrac{1}{8}\epsilon z V_0''' + O(\epsilon^2) \tag{6.10}$$
$$U_{-2} = -\tfrac{3}{8}V_0 V_0' - \tfrac{1}{16}V_0''' + O(\epsilon) \tag{6.11}$$

Substituting these quantities into the q-deformed KdV equations, we obtain their first-order form

$$\dot{V}_1 = \tfrac{1}{2}\epsilon z(6V_0 V_0' + V_0''') \tag{6.13}$$
$$\begin{aligned} \dot{V}_0 = &\tfrac{1}{4}(6V_0 V_0' + V_0''') - \tfrac{3}{4}V_0' V_1' - \tfrac{3}{4}V_0 V_1'' - \tfrac{1}{8}V_1'''' \\ &+ \tfrac{3}{2}\epsilon z V_0 V_0'' + \tfrac{1}{4}\epsilon z V_0'''' - \tfrac{1}{2}\epsilon V_0''' + \tfrac{3}{2}\epsilon z V_0'^2 \end{aligned} \tag{6.14}$$

where $\dot{V} = \partial V/\partial t$.

Now we expand V_0 and V_1 in ϵ

$$V_0 = X_0 + \epsilon X_2 \qquad V_1 = \epsilon X_1.$$

Equations (6.13)–(6.14) become

$$\dot{X}_0 = \tfrac{1}{4}(X_0''' + 6X_0X_0') \tag{6.15}$$

$$\dot{X}_1 = \tfrac{1}{2}z(X_0''' + 6X_0X_0') \tag{6.16}$$

and

$$\dot{X}_2 = \tfrac{1}{4}(X_2''' + 6X_0X_2' + 6X_0'X_2) - \tfrac{3}{4}X_0'X_1' - \tfrac{3}{4}X_0X_1'' - \tfrac{1}{8}X_1'''' + \tfrac{1}{4}zX_0'''' - \tfrac{1}{2}X_0''' + \tfrac{3}{2}zX_0'^2 - \tfrac{3}{2}zX_0X_0''. \tag{6.17}$$

The above second equation is total differential $\dot{X}_1 = 2z\dot{X}_0$, its solution is $X_1 = 2zX_0 + f(z)$. For convenience we only consider the case of $f(z) = 0$, we have finally

$$\dot{X}_2 = \tfrac{1}{4}(X_2''' + 6X_0X_2' + 6X_0'X_2) - \tfrac{3}{2}(X_0''' + 3X_0X_0'). \tag{6.18}$$

Equation (6.15) is just an ordinary KdV equation. The first order q-deformed modification X_2 can be solved from (6.18) and X_1 is given by $2zX_0$.

7. The conservation quantities of the first-order q-KdV equation

Up to the first order

$$\Gamma_q = 1 + \epsilon \tag{7.1}$$

and from equation (5.15)

$$\mathrm{res}L^{r/k} = (1+\epsilon)a_{r+1} - 2\epsilon z a_{r+2} + \mathrm{O}(\epsilon^2) \tag{7.2}$$

therefore the densities of conservation laws are

$$H_{l-1/2} = \mathrm{res}L^{l-1/2} = (1+\epsilon)R_l - 2\epsilon z S_l + \mathrm{O}(\epsilon^2). \tag{7.3}$$

Let us expand them in powers of the q-deformed infinitesimal parameter ϵ. Let

$$R_l = r_l + \epsilon p_l + \mathrm{O}(\epsilon^2) \tag{7.4}$$

$$S_l = h_l + \epsilon g_l + \mathrm{O}(\epsilon^2). \tag{7.5}$$

From the recursion relation (5.23) of R_l one has for the zero order of ϵ

$$h_l = -\tfrac{1}{2}r_l' \tag{7.6}$$

and for the first order

$$g_l = -\tfrac{1}{2}p_l' - zX_0r_l. \tag{7.7}$$

From the recursion relation (5.24) of S_l one has for the zero order

$$r'_{l+1} = \tfrac{1}{4}r'''_l + r'_l X_0 + \tfrac{1}{2}r_l X'_0 \tag{7.8}$$

and for the first order

$$p'_{l+1} = (-2zr_l X'_0 + p_l X_0 + r_l X_2 + \tfrac{1}{4}p''_l - r_l X_0)' + r_l(zX''_0 + \tfrac{3}{2}X'_0 - \tfrac{1}{2}X'_2) - \tfrac{1}{2}p_l X'_0. \tag{7.9}$$

Taking $r_0 = 1$, $p_0 = 0$ one obtains for the first three orders

$$r_1 = \tfrac{1}{2}X_0 \tag{7.10}$$

$$p_1 = -\tfrac{1}{2}X_0 - zX'_0 + \tfrac{1}{2}X_2 \tag{7.11}$$

$$r_2 = \tfrac{3}{8}X_0^2 + \tfrac{1}{8}X''_0 \tag{7.11}$$

$$p_2 = -\tfrac{3}{2}zX_0X'_0 + \tfrac{3}{4}X_0X_2 - \tfrac{3}{4}X_0^2 + \tfrac{1}{8}X''_2 - \tfrac{5}{8}X''_0 - \tfrac{1}{4}zX'''_0 \tag{7.12}$$

$$r_3 = \tfrac{5}{32}X_0'^2 + \tfrac{5}{16}X_0X''_0 + \tfrac{1}{32}X''''_0 + \tfrac{5}{16}X_0^3 \tag{7.13}$$

$$p_3 = -\tfrac{15}{8}zX_0^2X'_0 - \tfrac{5}{4}zX'_0X''_0 + \tfrac{15}{16}X_0^2X_2 - \tfrac{15}{16}X_0^3 + \tfrac{5}{16}X_0X''_2 - \tfrac{15}{8}X_0X''_0 - \tfrac{5}{8}zX_0X'''_0 + \tfrac{5}{16}X''_0X_2 - \tfrac{15}{16}X_0'^2 + \tfrac{5}{16}X'_0X'_2 + \tfrac{1}{32}X''''_2 - \tfrac{9}{32}X''''_0 - \tfrac{1}{16}zX_0^{(5)}. \tag{7.14}$$

We see that r_l are just the ordinary Gelfand–Dikii potentials. As a first-order approximation,

$$-2\epsilon zS_l = \epsilon zR'_l \tag{7.15}$$

so that

$$H_{l-1/2} = (1+\epsilon)R_l + \epsilon zR'_l = R_l + \epsilon(zR_l)'. \tag{7.16}$$

We learn the $H_{l-1/2}$ differs from R_l only by a total differential. From (7.3) we have

$$H_{l-1/2} = r_l + \epsilon(r_l + p_l + zr'_l). \tag{7.17}$$

For the first three orders we obtain

$$H_{1/2} = \tfrac{1}{2}V_0 - \tfrac{1}{2}\epsilon zV'_0 + O(\epsilon^2) \tag{7.18}$$

$$H_{3/2} = \tfrac{3}{8}V_0^2 + \tfrac{1}{8}V''_0 - \tfrac{3}{8}\epsilon V_0^2 - \tfrac{1}{2}\epsilon V''_0 - \tfrac{3}{4}\epsilon zV_0V'_0 - \tfrac{1}{8}\epsilon zV'''_0 + O(\epsilon^2) \tag{7.19}$$

$$H_{5/2} = \tfrac{5}{32}V_0'^2 + \tfrac{5}{16}V_0V''_0 + \tfrac{1}{32}V''''_0 + \tfrac{5}{16}V_0^3 - \epsilon(\tfrac{15}{16}zV_0^2V'_0 + \tfrac{5}{8}zV'_0V''_0 + \tfrac{5}{8}V_0^3 + \tfrac{25}{16}V_0V''_0 + \tfrac{5}{16}zV_0V'''_0 + \tfrac{25}{32}V_0'^2 + \tfrac{1}{4}V''''_0 + \tfrac{1}{32}zV_0^{(5)}) + O(\epsilon^2). \tag{7.20}$$

Using the definition of variation,

$$\frac{\delta H}{\delta V_0} = \sum_{n=0}^{N}(-1)^n\left(\frac{\partial H}{\partial V_0^{(n)}}\right)^{(n)} \tag{7.21}$$

one can directly verify that the results for the first three orders satisfied the above relation

$$\frac{\delta H_{l+1/2}}{\delta V_0} = (l+\tfrac{1}{2})(H_{l-1/2} + \epsilon zR'_l). \tag{7.22}$$

Due to

$$H_{l-1/2} = R_l + \epsilon(zR_l)'$$

we have

$$\frac{\delta H_{l-1/2}}{\delta V_0} = \frac{\delta R_l}{\delta V_0}. \tag{7.23}$$

From (3.15), (3.11) and (5.17) one has

$$\begin{aligned}\frac{\partial V_0}{\partial t} &= 4\epsilon z S_2' + (2-2\epsilon)(R_2' + \epsilon(2R_2' + 3zR_2'')) \\ &= 2(1-2\epsilon)((1+\epsilon)R_2 + 2\epsilon z R_2').\end{aligned} \tag{7.24}$$

One can rewrite the equation of motion as

$$\frac{\partial V_0}{\partial t} = \left(\frac{\delta}{\delta V_0}[\tfrac{4}{5}(1-2\epsilon)H_{5/2}]\right)' + \mathrm{O}(\epsilon^2). \tag{7.25}$$

The conservation quantities are

$$C_{l-1/2} = \int \mathrm{d}x\, H_{l-1/2} = \int \mathrm{d}x\, R_l. \tag{7.26}$$

Of these, the first three are

$$C_{1/2} = \int \mathrm{d}x\, \tfrac{1}{2}(1+\epsilon)V_0 \tag{7.27}$$

$$C_{3/2} = \int \mathrm{d}x\, \tfrac{3}{8}V_0^2 \tag{7.28}$$

$$C_{5/2} = \int \mathrm{d}x\, \tfrac{5}{32}(-V_0'^2 + 2V_0^3(1-\epsilon)). \tag{7.29}$$

8. Discussion

The q-deformed KdV equations (3.14-15) are in fact non-linear integrable evolution equations with q-differences. Their Lax pair structure and the existence of their infinitely many conservation laws are two strong indications that these systems are completely integrable. In order to prove reliably that the q-deformed KdV hierarchies are completely integrable, however, we must find their Poisson brackets and prove that these infinitely many conservation laws are in involution from each other. If we find the Poisson structure of the q-deformed KdV hierarchies, we shall know the true q-deformed Virasoro algebra and even the q-deformed W algebras. This is a very tempting and yet not a light problem. We are working on this subject matter. Therefore our investigation is only preliminary and a considerable number of new interesting problems are waiting to be studied. For example, how to find their solutions, how to find their applications in mathematics and physics.

Acknowledgments

This work was supported in part by the Foundation of National Education Committee of China and in part by the National Natural Science Foundation of China. The author would like to thank B Y Hou, B H Zhao, Y B Dai, H Y Guo and C S Huang for useful discussions.

Appendix A

Proposition A. Operator Q is equal to operator $\widetilde{Q}$.

Proof 1.

$$\widetilde{Q}f(z) = \left(\widetilde{Q}(\sum_n c_n(z-z_0)^n)\widetilde{Q}^{-1}\right)\cdot(\widetilde{Q}\cdot 1)$$
$$= \left(\sum_n c_n(\widetilde{Q}z\widetilde{Q}^{-1}-z_0)^n\right)\cdot(\widetilde{Q}\cdot 1)$$
$$= \left(\sum_n c_n(zq-z_0)^n\right)\cdot 1 = f(zq) = Qf(z).$$

Proof 2. It is very interesting and useful to give a direct proof that the expansion

$$Q = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n\epsilon^n z^n\partial^n \tag{2.4}$$

is equal to the expansion

$$\widetilde{Q} = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n\left(\sum_{m=1}^{\infty}\frac{1}{m}\epsilon^m\right)^n (z\partial)^n. \tag{2.6}$$

Letting

$$z^n\partial^n = \sum_{k=0}^{n-1}(-1)^k a_k^{(n)}(z\partial)^{n-k} \tag{A.1}$$

and using $z^n\partial = \partial z^n - nz^{n-1}$, we get the recursion formula

$$a_i^{(n)} = a_i^{(n-1)} + (n+1)\, a_{i-1}^{(n-1)} \tag{A.2}$$

and then

$$a_m^{(n)} = \sum_{1=i_1<i_2<\cdots<i_m=n-1}\ \prod_{l=1}^{m} i_l. \tag{A.3}$$

Therefore Q can be expressed as

$$Q = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n\epsilon^n\left(\sum_{i=1}^{\infty}\epsilon^i\ b_i^{(n)}\right)(z\partial)^n \tag{A.4}$$

where

$$b_i^{(n)} = \frac{n!}{(i+n)!}\, a_i^{(i+n)}. \tag{A.5}$$

Using the recursion formula (A.2), we obtain

$$b_i^{(n)} = \frac{n}{n+i}(b_0^{(n-1)} + b_1^{(n-1)} + \ldots + b_i^{(n-1)}) \tag{A.6}$$

$$= n! \sum_{0=i_n \leqslant i_{n-1} \leqslant \cdots \leqslant i_1 \leqslant i_0 = i} \prod_{k=0}^{n-1} (n-k-i_k)^{-1}. \tag{A.7}$$

On the other hand, if we assume

$$\left(\sum_{m=1}^{\infty} \frac{1}{m}\epsilon^{m-1}\right)^n = \sum_{i=0}^{\infty} \tilde{b}_i^{(n)}\, \epsilon^i \tag{A.8}$$

we can rewrite $\widetilde{Q}$ as

$$\widetilde{Q} = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \epsilon^n \left(\sum_{i=1}^{\infty} \epsilon^i\, \tilde{b}_i^{(n)}\right)(z\partial)^n. \tag{A.9}$$

Using $\ln q \cdot (\ln q)^n = (\ln q)^{(n+1)}$, we get the recursion formula

$$\tilde{b}_i^{(n)} = \sum_{k=0}^{i} \frac{1}{i-k+1} \tilde{b}_k^{(n-1)} \tag{A.10}$$

and finally obtain

$$\tilde{b}_i^{(n)} = \sum_{0=i_n \leqslant i_{n-1} \leqslant \cdots \leqslant i_1 \leqslant i_0 = i} \prod_{k=0}^{n-1} (i_k - i_{k+1} + 1)^{-1}. \tag{A.11}$$

Using repeatedly the decomposing relation

$$\frac{1}{i_k + n - k + 1}\left(\frac{1}{i_k - i_{k+1} + 1} + \frac{1}{n - k + i_{k+1}}\right) = \frac{1}{(i_k - i_{k+1} + 1)(n - k + i_{k+1})}$$

and the recursion method, one can prove $\tilde{b}_i^{(n)} = b_i^{(n)}$, therefore $\widetilde{Q} = Q$.

Appendix B

Lemma. If $dL/dt = [A, L]$, then $d(L)^{r/k}/dt = [A, L^{r/k}]$.

Proof [20]. We set $M = L^{r/k}$. It is given that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - A, L\right] = 0. \tag{B.1}$$

It is necessary to prove that $[\mathrm{d}/\mathrm{d}t - A, M] = 0$. Since $M^k = L^r$, it follows from (B.1) that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - A, M^k\right] = 0. \tag{B.2}$$

On the other hand,

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - A, M^k\right] = \sum_{l=1}^{k} M^{l-1}\left[\frac{\mathrm{d}}{\mathrm{d}t} - A, M\right]M^{k-l}. \tag{B.3}$$

It is easy to see that the leading coefficient on the right side of (B.3) is larger than the leading coefficient of $[\mathrm{d}/\mathrm{d}t - A, M] = 0$ by a factor of k. Therefore, the assumption that $[\mathrm{d}/\mathrm{d}t - A, M] \neq 0$ contradicts (B.2).

Proposition B. If

$$\frac{\partial L}{\partial t} = [M_+, L] \qquad M = \sum c_l L^{l/k} \tag{5.3}$$

and

$$\frac{\partial L}{\partial \tau} = [\tilde{M}_+, L] \qquad \tilde{M} = \sum \tilde{c}_l L^{l/k} \tag{5.4}$$

then

$$\frac{\partial^2 L}{\partial t \partial \tau} = \frac{\partial^2 L}{\partial \tau \partial t}. \tag{5.5}$$

Proof [20]. We have

$$\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial t}\right) = \frac{\partial}{\partial \tau}[M_+, L] = \left[\frac{\partial M_+}{\partial \tau}, L\right] + \left[M_+, \frac{\partial L}{\partial \tau}\right]. \tag{B.4}$$

According to Lemma

$$\frac{\partial M_+}{\partial \tau} = [\tilde{M}_+, M]_+. \tag{B.5}$$

Thus,

$$\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial t}\right) = [[\tilde{M}_+, M]_+, L] + [M_+, [\tilde{M}_+, L]]. \tag{B.6a}$$

Similarly,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \tau}\right) = [[M_+, \tilde{M}]_+, L] + [\tilde{M}_+, [M_+, L]]. \tag{B.6b}$$

Using the Jacobi identity, we obtain

$$\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \tau}\right) = [[\tilde{M}_+, M]_+ - [M_+, \tilde{M}]_+ + [M_+, \tilde{M}_+], L] \tag{B.7}$$

but

$$[\tilde{M}_+, M]_+ = [M, \tilde{M}_-]_+ = [M_+, \tilde{M}_-]_+ \tag{B.8}$$

so that

$$[\tilde{M}_+, M]_+ - [M_+, \tilde{M}]_+ + [M_+, \tilde{M}_+] = 0. \tag{B.9}$$

References

[1] As a review, see, for example,
Drazin P G and Johnson R S 1989 *Soliton: an Introduction* (Cambridge: Cambridge University Press)
[2] Gross D J and Migdal A A 1990 *Nucl. Phys.* B **340** 333
[3] Brezin E and Kazakov V 1990 *Phys. Lett.* **236B** 144
[4] Knizhnik V, Polyakov A and Zamolodchikov A 1988 *Mod. Phys. Lett.* A **3** 819
[5] Dijikgraaf R and Witten E 1990 *Preprint* IASSNS-HEP-90/18
[6] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[7] Zamolodchikov A B 1985 *Theor. Math. Phys.* **65** 1205 Zhang D H 1989 *Phys. Lett.* **232B** 323
[8] Drinfeld V G 1986 *Proc. Intern. Congress of Mathematicians (Berkeley)* vol 1 (New York: Academic) p 793
[9] Baxter R 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic Press)
[10] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[11] Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
[12] Bernard D and LeClair A 1989 *Phys. Lett.* **227B** 417
Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415
[13] Gray R W and Nelson C A 1990 *J. Phys. A: Math. Gen.* **23** L945
[14] Aizawa N and Sato H 1990 *Preprint* HUPD-9012
[15] Curtright T and Zachos C 1990 *Phys. Lett.* **243B** 237
[16] Chaichian M, Kulish P and Lukierski J 1990 *Phys. Lett.* **237B** 401
[17] Chaichian M, Popowicz Z and Presnajder P 1990 *Phys. Lett.* **249B** 63
[18] Jackson F H 1910 *Quart. J. Math.* **41** 193
[19] Lax P P 1969 *Comm. Pure Appl. Math.* **21** 467
[20] Drinfeld V G and Sokolov V V 1985 *J. Sov. Math.* **30** 1975